Lectured By Dr. H. K. Yadav.
Deptt. of Mathematics
Marwari College, Darbhanga. Date- 03-06-2020
Set theory. B.Sc-part-I(H), paper-I

Q. 1. Define countable and uncountable sets and prove that the set of rational numbers is countable.

Soln. Countable and uncountable sets:—

Let X be any set can be mapped one-one onto the set $\{1, \dots, n\}$, where $n \in N$, then X is called a finite set. We also say that X has n elements.

In this case, if $X = \{1, \dots, n\}$ is established by function f and $1 \le k \le n$, then under f, the image of $k \in \{1, \dots, n\}$ is denoted as f_k. Hence, we write $X = \{f_1, \dots, f_n\}$. The empty set ϕ is taken as finite.

Here, the ordered n-tuple $\langle f_1, \dots, f_n \rangle$ is merely a permutation of X. In all, there are $\underline{n}$ permutations.

The members of $N = \{1, 2, 3, 4, \dots, n, \dots\}$ are called finite natural numbers.

A set X which is not finite is called an infinite set. From the above discussion, it will be clear that X will be infinite only when there is no one-one map of X onto $\{1, \dots, n\}$ for any n.

Proof: Let $A_n = \{\frac{0}{n}, -\frac{1}{n}, \frac{1}{n}, -\frac{2}{n}, \frac{2}{n}, \dots\}$

So that A_n is the set of all those rationals whose denominator is $n \in N$. Here, it is clear that

$$f : N \to A_n$$

defined by $f(x) = \begin{cases} \frac{x-1}{2n}, & \text{when } x \text{ is odd.} \\ -\frac{x}{2n}, & \text{when } x \text{ is even} \end{cases}$

is one-one onto mapping. Thus each A_n is countable.

Now, $Q = \bigcup A_n$ is a countable union of countable sets and hence it is countable.

Thus, the set of rational numbers is countable.

2114 B.Sc - part - I (III), paper - I, Date 04-06-2020

Set theory

Q. 2. Prove that the set of all irrational numbers is uncountable.

Proof :- Since the set R of all real numbers and the set Q of rational numbers is countable. then if B is a countable subset of an uncountable set A, then $A-B$ is uncountable. ~~It follows that~~ The complement of the set of rationals relative to the set of real numbers i.e. $R-Q$ of irrational numbers is uncountable.

Q. 3. Prove that the set A of algebraic numbers is countable enumerable.

Proof: Let $P = \cup \{P_n(x) : P_n(x) = 0,\ n \in N\}$, where $P_n(x)$ stands for a polynomial of degree n with integral coefficients, is enumerable. When

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_mx^m$$

with integral coefficients is enumerable.

Since, an algebraic number is the root of a polynomial $p(x) = 0$ with integral coefficients. Let us consider $A_n = \{x : x \text{ is a solution of } P_n(x) = 0\}$

Since, a polynomial of degree n has at most n roots and hence A_n is finite.

Now, $A = \cup \{A_n : n \in N\}$ is the set of algebraic numbers and it is union of countable sets and hence A, the set of all algebraic numbers is countable.

Q. 4. Prove that the set of all irrational numbers is uncountable.

Proof: Since the set R of all real numbers ~~and~~ is uncountable because of the set of unit interval $[0, 1]$ is uncountable. and the set of rational numbers is countable. when the set of all positive rationals is countable. Then if B is a countable subset of an uncountable set A, then $A-B$ is uncountable. The complement of the set of rationals relative to the set of real numbers. i.e. $R-Q$ of irrational numbers is uncountable.